ON A PROBLEM OF CHAPLYGIN

(OB ODNOI ZADACHE CHAPLYGINA)

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In the article "On the Theory of Motion of a Nonholonomic System. Theorem on a Reducing Multiplier", Chaplygin [1] promulgated a certain integral variational principle for a class of nonholonomic systems.

Let ξ, η, q, q_1 be the Lagrange coordinates of a material system which satisfy two nonholonomic equations

$$\xi' = aq' + a_1 q_1', \qquad \eta' = bq' + b_1 q_1'$$

whose coefficients a, a_1, b, b_1 do not depend on ξ, η and time t. We assume also that the kinetic energy T, and the force function U are independent of ξ, η and t.

Through the introduction of a new independent variable τ by means of the equation $Ndt = d\tau$, Chaplygin selected a function $N(q, q_1)$ (the so-called "reducing multiplier") in such a way that the equations of motion of a system in the space q, q_1, τ would take on the canonical form. For this, the function N had to satisfy, identically in the impulses $p = \partial T^*/\partial \dot{q}$, $p_1 = \partial T^*/\partial q_1$, the equation

$$NS - \frac{\partial T^*}{\partial \dot{q}_1} \frac{1}{N} \frac{\partial N}{\partial q} + \frac{\partial T^*}{\partial \dot{q}} \frac{1}{N} \frac{\partial N}{\partial q_1} = 0 \tag{0.1}$$

Here

$$\dot{q} = \frac{dq}{d\tau}, \qquad \dot{q}_1 = \frac{dq_1}{d\tau}, \qquad S = \frac{\partial T}{\partial \xi'} \left(\frac{\partial a}{\partial q_1} - \frac{\partial a_1}{\partial q} \right) + \frac{\partial T}{\partial \eta'} \left(\frac{\partial b}{\partial q_1} - \frac{\partial b_1}{\partial q} \right)$$

and $T^*(q, q_1, \dot{q}, \dot{q}_1)$ is the reduced expression for the kinetic energy. For such a function N in the space q, q_1, τ, Hamilton's principle is valid:

$$\delta \int_0^{\tau_1} (T^* + U)\, d\tau = 0$$

Returning now to the time t, Chaplygin obtains the theorem: if the reducing multiplier N given by (0.1) exists, then the real motion of a system with freely varying (within certain limits) parameters q and q_1 must be such that the variation of the integral

$$\delta \int_0^{t_1} (T^{**} + U) N\, dt = 0$$

vanishes under the condition that

$$\tau_1 = \int_0^{t_1} N\, dt$$

be constant.

Here $T^{**}(q, q_1, q', q_1')$ is the reduced expression for the kinetic energy.

Chaplygin's theorem has a dual meaning. Firstly, it establishes that the real motions of certain nonholonomic systems in space of the free coordinates q, q_1 satisfy certain definite extremal conditions. Secondly, it permits the use of the Hamilton-Jacobi method of integration for the determination of these real motions.

Below, an attempt will be made to extend Chaplygin's results to a broader class of nonholonomic systems and to the space of all the Lagrange variables.

1. Suppose that we are given some material system. Let $q_1, \ldots, q_n$ be its Lagrange coordinates restricted by the nonholonomic equations of constraint

$$\omega_\beta = q_\beta' + \sum_{r=1}^{n-m} a_{\beta, m+r} q_{m+r}' + a_\beta = 0 \qquad (\beta = 1, \ldots, m) \tag{1.1}$$

whose coefficients $a_{\beta, m+r}$, a_β depend on all the coordinates of the points $q_1, \ldots, q_n$ and the time t.

The motion $q_s = \varphi_s(t)$ $(s = 1, \ldots, n)$ of a material system is said to be kinematically admissible if the functions $\varphi_s(t)$ satisfy identically the equations of constraint.

Let

$$F = \frac{1}{2} \sum_{s, k=1}^{n} b_{sk} q_s' q_k' + \sum_{s=1}^{n} c_s q_s' + P \tag{1.2}$$

Here b_{ks}, c_s and P are some functions of the coordinates and time. Let us find out for what coefficients of the function F, and for what boundary conditions, the set of the real motions of the system coincides with the set of extremals of the conditional variational problem

$$\delta \int_{t_0}^{t_1} F\, dt = 0 \quad \text{when } \omega_\beta = 0 \tag{1.3}$$

2. We will suppose that the real motions of the system are determined by the variational problem (1.3). Now we must specify what necessary conditions must be satisfied in this case by the boundary conditions of the problem (1.3).

Let $\lambda_\beta(t)$ be the Lagrange multipliers for the problem (1.3). By separating in its equation the Euler terms with q_k'' ($k = 1, \ldots, n$), we obtain

$$\sum_{k=1}^{n} \frac{\partial^2 F}{\partial q_s' \partial q_k'} q_k'' + \sum_{k=1}^{n} \frac{\partial^2 F}{\partial q_s' \partial q_k} q_k' + \frac{\partial^2 F}{\partial q_s' \partial t} - \frac{\partial F}{\partial q_s} +$$
$$+ \sum_{\beta=1}^{m} \lambda_\beta' a_{\beta s} + \sum_{\beta=1}^{m} \lambda_\beta \left(a'_{\beta s} - \frac{\partial \omega_\beta}{\partial q_s}\right) = 0 \qquad (s = 1, \ldots, n) \tag{2.1}$$

Here

$$a_{\beta s} = \begin{cases} 1 & \text{if } s = \beta \\ 0 & \text{if } m > s \neq \beta \\ a_{\beta, m+r} & \text{if } s = m + r \end{cases}$$

The equations of constraint yield

$$\sum_{k=1}^{n} a_{\beta k} q_k'' + \sum_{k=1}^{n} a'_{\beta k} q'_k + a_\beta' = 0 \qquad (\beta = 1, \ldots, m) \tag{2.2}$$

We exclude the singular case, and assume that

$$\Delta = \begin{vmatrix} \| \partial^2 F_k / \partial q_s' \partial q_k' \| & \| a_{\beta s} \| \\ \| a_{\beta k} \| & \| 0 \| \end{vmatrix} \neq 0 \tag{2.3}$$

Then we obtain from (2.1) and (2.2) the unique values

$$q_k'' = \frac{\Delta_k}{\Delta} = Q_k(q, q', t, \lambda) \qquad (k = 1, \ldots, n)$$
$$\lambda_\beta' = \frac{\Delta_{n+\beta}}{\Delta} = \Lambda_{n+\beta}(q, q', t, \lambda) \qquad (\beta = 1, \ldots, m) \tag{2.4}$$

Here Δ_k and $\Delta_{n+\beta}$ are obtained from the determinant Δ by replacing the elements of the kth, or the $(n + \beta)$th columns by the free terms of Equations (2.1) and (2.2).

In mechanics, one has the "generalized principle of inertia" (in the terminology of Poincaré): the accelerations of the points of a mechanical system can be uniquely determined at any instant of time if one knows the forces, the constraints, the coordinates, and the velocities of the points at the given instant of time.*

By hypothesis, the real motions of the system are determined by the variational problem (1.3). In the light of the above principle, let us consider Equation (2.4). The "generalized principle of inertia" requires that the multipliers λ_β, which enter into the function Q_k, be determined completely by the values t, q, q' (at the given instant), independently of the initial conditions. From the first m integrals of the equations for the extremals, it is possible to find λ_β as a function of t, q, q', and of m constants k_γ of integration. The constants k_γ are determined by means of the boundary conditions of the problem, and they will, therefore, differ for different extremals.

The "generalized principle of inertia" can be satisfied in two cases only:

First, if the multipliers λ_β have the same constant values k_γ on all extremals. The the values

$$\lambda_\beta = \lambda_\beta (t, q, q')$$

are determined by the first integrals of the equations of the extremals.

Second, if for all kinematically admissible motions** the multipliers λ_β do not enter into the functions Q_k. This will happen only if the equations of constraint are integrable [2].

Thus, in order that the real motions of a nonholonomic system may be described by the variational problem (1.3), its boundary conditions must satisfy special requirements. They must be such that the multipliers λ_β, determined by means of the first integrals of the equations of the extremals, have the same constants k_γ of integration on all extremals.

We note, for example, that the boundary conditions of Hamilton's principle do not satisfy this condition.

Let us show under what boundary conditions of the variational problems (1.3) the indicated necessary condition is satisfied.

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* This implies, in particular, that condition (2.3) is satisfied.

** Since the real motions are assumed to be unknown, all assertions are formulated for arbitrary kinematically admissible motions.

Let us consider a material system in which the function of Lagrange, and the equation of constraint (1.1) do not depend on the coordinates $q_1, \ldots, q_m$.

We will put the same requirement on the function F defined by (1.2).

Then Euler's equation (2.1) will have m "cyclic integrals"

$$\frac{\partial}{\partial q_\gamma'}\left(F + \sum_{\beta=1}^{m} \lambda_\beta \omega_\beta\right) = k_\gamma \qquad (\gamma = 1, \ldots, m) \tag{2.5}$$

which, in view of (1.2) and (1.3), can be rewritten in the form

$$\sum_{s=1}^{n} b_{\gamma s} q_s' + c_\gamma + \lambda_\gamma = k_\gamma \tag{2.6}$$

Let us select the following boundary condition of the variational problem (1.3)

$$q_s(t_0) = q_{s0}, \qquad q_{m+r}(t_1) = q_{m+r,\,1} \qquad (s = 1, \ldots, n;\ r = 1, \ldots, n-m) \tag{2.7}$$

for arbitrary coordinates q_β ($\beta = 1, \ldots, m$) of the finite point M_1. Then the next conditions of transversality

$$\left[\frac{\partial}{\partial q_\gamma'}\left(F + \sum_{\beta=1}^{m} \lambda_\beta \omega_\beta\right)\right]_{M_1} = 0 \qquad (\gamma = 1, \ldots, m) \tag{2.8}$$

must be satisfied at all finite points M_1 of the extremals. The conditions mentioned yield definite values for the arbitrary constants k_γ from (2.5), namely, $k_\gamma = 0$. Finally, we have

$$\lambda_\gamma = -\sum_{s=1}^{n} b_{\gamma s} q_s' - c_\gamma \qquad (\gamma = 1, \ldots, m) \tag{2.9}$$

along all extremals.

Let us now consider a different class of mechanical systems. Suppose that the Lagrange coordinates of a system are restricted by one nonholonomic equation of constraint

$$\omega = q_1' + \sum_{r=1}^{n-1} a_{1+r} q_{1+r}' + a = 0 \qquad (a \neq 0) \tag{2.10}$$

Let us assume that the Lagrange functions of the system, and of the equation of constraint do not depend on the time t. We make the same assumption for the function F. Then Euler's equations of the problem will have a quadratic first integral

$$F - \sum_{s=1}^{n} \frac{\partial F}{\partial q_s'} q_s' + \lambda a = k \tag{2.11}$$

Let us leave free the upper limit of the integral (1.3) but keep fixed the coordinates of the boundary points

$$q_s(t_0) = q_{s0}, \qquad q_s(t) = q_{s1} \qquad (s = 1, \ldots, n) \tag{2.12}$$

For the chosen boundary conditions at the finite point M_1, we have the following condition of transversality

$$\left[F - \sum_{s=1}^{n} \frac{\partial F}{\partial q_s'} q_s' + \lambda a\right]_{M_1} = 0 \tag{2.13}$$

which yields a definite value for the arbitrary constant k of (2.11), namely, $k = 0$. Finally, we find that along all extremals

$$\lambda = \frac{1}{a}\left(\sum_{s=1}^{n} \frac{\partial F}{\partial q_s'} q_s' - F\right) \qquad (a \neq 0) \tag{2.14}$$

Let us now substitute (2.9) or (2.14) into the first equations (2.4). We then obtain a system of the form

$$q_k'' = R_k(q, q', t) \qquad (k = 1, \ldots, n) \tag{2.15}$$

where the R_k are some functions. It is obvious that the substituted values λ_β, which were determined by means of the first integrals of Euler's equations, will reduce the second equations (2.4) to identities. In Equations (2.15), the variables enter through the coefficients of the function F defined by (1.2).

3. By hypothesis, the set of extremals, the solutions of Equations (2.15), coincides with the set of real motions, the solutions of the given equations of motion

$$q_k'' = \Phi_k(q, q', t) \qquad (k = 1, \ldots, n) \tag{3.1}$$

It is obvious that for this to be true it is necessary and sufficient that the functions Φ_k and R_k must coincide because of the equations of constraint.* The requirement of the "equivalence of Equations (2.15) and (3.1)", imposes certain conditions on the coefficients of the function F (1.2).

Let us write down the condition of the equivalence of equations. We will denote by

$$R_{k0}, \quad \Phi_{k0}, \quad R_{k,m+r}, \quad \Phi_{k,m+r}, \quad R_{k,m+r,m+\rho}, \quad \Phi_{k,m+r,m+\rho}, \quad R_{k,m+r,m+\rho,m+\tau}$$
$$(k = 1, \ldots, n; \; r, \rho, \tau = 1, \ldots, n-m)$$

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* See last footnote.

the coefficients of the terms of the zeroth, first, second, and third orders relative to the independent velocities, in the functions R_k and Φ_k after we have eliminated from them the dependent velocities q_β' with the aid of the equations of constraint.

The conditions of equivalence are

$$R_{k0} = \Phi_{k0}, \quad R_{k,\,m+r} = \Phi_{k,\,m+r}, \quad R_{k,\,m+r,\,m+\rho} = \Phi_{k,\,m+r,\,m+\rho}$$
$$(k = 1, \ldots, n;\ r, \rho = 1, \ldots, n-m) \tag{3.2}$$

Since R_k and Φ_k, for the first class systems, do not involve terms of the third order in the velocities, the equations terminate with them. For the second class of systems with the multiplier λ given by (2.14), one must consider also the equations

$$R_{k,\,1+r,\,1+\rho,\,1+\tau} = 0 \qquad (k = 1, \ldots, n;\ r, \rho, \tau = 1, \ldots, n-1) \tag{3.3}$$

We note that for the equivalence of the equations for all $k = 1, \ldots, n$, it is sufficient to have them equivalent for $k = m + 1, \ldots, n$. Indeed, suppose that the functions R_{m+r} and Φ_{m+r} ($r = 1, \ldots, n - m$) coincide in view of the equations of constraint. Substituting these functions into the equations of constraint, we establish the equality of the remaining functions R_β and Φ_β ($\beta = 1, \ldots, m$).

It is for this reason that in the sequel, we shall mean by the term "equations of equivalence" the last equations of (3.2) and (3.3) with $k = m + 1, \ldots, n$.

Remark. Sometimes it is possible to introduce into the coefficients of the function F, such constants that some of the terms of the equations $R_k - \Phi_k = 0$ vanish in view of the first integrals of the equations of motion. This simplifies the equations of equivalence. A similar method will be used in Section 4.

Let us first consider the first class of nonholonomic systems.

Equations (3.2) are first order, linear, nonhomogeneous, partial differential equations in the unknown functions b_{sk}, c_s and P of the coordinates $q_{m+1}, \ldots, q_n$ and time t. These functions are the coefficients of the function

$$F = \frac{1}{2} \sum_{s,\,k=1}^{n} b_{sk} q_s' q_k' + \sum_{s=1}^{n} c_s q_s' + P$$

We thus obtain the following theorem.

Theorem 3.1. Suppose that we are given a material system with nonholonomic constraints (1.1) that are independent of the cyclic coordinates q_β ($\beta = 1, \ldots, m$), and suppose that there exists for it a solution of the equations of equivalence (3.2). Then every real motion in the class of all kinematically admissible motions of this system which satisfy the boundary conditions

$$q_s(t_0) = q_{s0}, \qquad q_{m+r}(t_1) = q_{m+r,1} \qquad (s = 1, \ldots, n;\ r = 1, \ldots, n-m)$$

is such that

$$\delta \int_{t_0}^{t_1} F\,dt = 0$$

Thus, we see that a necessary and sufficient condition, for the existence of the variational problem (1.3) for the equations of motion of a nonholonomic system, is the existence of a solution of the equations of equivalence. It is important to note that for the construction of the variational problem it is sufficient to know some particular solution of these equations.

The Theorem 3.1 asserts that the real motions of nonholonomic systems can possess in the space of all coordinates $\{q_s\}$ a definite extremal property among all kinematically admissible motions. Furthermore, the theorem shows for what particular variational problem the real motions of the system serve as extremals.

We note that the extremals of the variational problem that describes the motion of a nonholonomic system in the space of all coordinates $\{q_s\}$, do not form a field.

4. In the case when the equations of equivalence admit a particular solution $b_{\beta k} = b_{k\beta} = c_\beta = 0$ ($\beta = 1, \ldots, m$; $k = 1, \ldots, n$), the function F (1.2) depends only on the independent variables $q'_{m+1}, \ldots, q_n$. To this function F there corresponds the variational problem which describes the motion of a nonholonomic system in the space $\{q_{m+r}\}$. If one knows this problem, one can use, for the solution of the last $n - m$ equations of motion of the nonholonomic system, all methods of integration known for holonomic systems; in particular, the Hamilton-Jacobi method. The change of the coordinates $q_1, \ldots, q_n$ of the system is then found by quadratures of the equations of constraint.

This case generalizes Chaplygin's theorem mentioned in the introduction. Let us alter the formulation of this theorem. We will use the notation given in the introduction. The variational problem of Chaplygin can be classified with the isoperimetric problems on the extremum of the integral

$$\int_{t_0}^{t_1} (T^{**} + U) N \, dt \tag{4.1}$$

under the condition

$$\tau_1 = \int_{t_0}^{t_1} N \, dt = \text{const} \tag{4.2}$$

Here $T^{**}(q, q_1, q', q_1')$ is the reduced expression for the kinetic energy after the exclusion of the dependent velocities. We select the constant τ_1 in the condition (4.2) so that Euler's equations, for the isoperimetric problem with the function $N(q, q_1)$ satisfying condition (0.1) for the reducing multiplier

$$NS - \frac{\partial T^{**}}{\partial q_1'} \frac{\partial N}{\partial q} + \frac{\partial T^{**}}{\partial q'} \frac{\partial N}{\partial q_1} = 0 \tag{4.3}$$

may coincide with Chaplygin's equations of motion

$$\frac{d}{dt} \frac{\partial T^{**}}{\partial q'} - \frac{\partial T^{**}}{\partial q} - \frac{\partial U}{\partial q} = q_1' S, \quad \frac{d}{dt} \frac{\partial T^{**}}{d q_1'} - \frac{\partial T^{**}}{\partial q_1} - \frac{\partial U}{\partial q_1} = - q' S \tag{4.4}$$

Let us write out Euler's equations of the isoperimetric problem (4.1) when μ = const is Lagrange's multiplier of the condition (4.2),

$$\begin{aligned} \frac{d}{dt} \frac{\partial T^{**}}{\partial q'} - \frac{\partial T^{**}}{\partial q} - \frac{\partial U}{\partial q} &= \frac{1}{N} \frac{\partial N}{\partial q} (T^{**} + U + \mu) - \frac{\partial T^{**}}{\partial q'} \frac{1}{N} \left(\frac{\partial N}{\partial q} q' + \frac{\partial N}{\partial q_1} q_1' \right) \\ \frac{d}{dt} \frac{\partial T^{**}}{\partial q_1'} - \frac{\partial T^{**}}{\partial q_1} - \frac{\partial U}{\partial q_1} &= \frac{1}{N} \frac{\partial N}{\partial q} (T^{**} + U + \mu) - \frac{\partial T^{**}}{\partial q_1'} \frac{1}{N} \left(\frac{\partial N}{\partial q} q' + \frac{\partial N}{\partial q_1} q_1' \right) \end{aligned} \tag{4.5}$$

Comparing Equations (4.4) and (4.5), we obtain

$$\begin{aligned} q_1' NS + \frac{\partial N}{\partial q} \frac{\partial T^{**}}{\partial q'} q' + \frac{\partial N}{\partial q_1} \frac{\partial T^{**}}{\partial q'} q_1' - \frac{\partial N}{\partial q} T^{**} - \frac{\partial N}{\partial q} (U + \mu) = 0 \\ - q' NS + \frac{\partial N}{\partial q} \frac{\partial T^{**}}{\partial q_1'} q' + \frac{\partial N}{\partial q_1} \frac{\partial T^{**}}{\partial q_1'} q_1' - \frac{\partial N}{\partial q_1} T^{**} - \frac{\partial N}{\partial q_1} (U + \mu) = 0 \end{aligned} \tag{4.6}$$

Equations (4.6) go over into

$$\begin{aligned} q_1' \left[NS + \frac{\partial T^{**}}{\partial q'} \frac{\partial N}{\partial q_1} - \frac{\partial T^{**}}{\partial q'_1} \frac{\partial N}{\partial q} \right] + \frac{\partial N}{\partial q} (T^{**} - U - \mu) = 0 \\ - q_1' \left[NS + \frac{\partial T^{**}}{\partial q'} \frac{\partial N}{\partial q_1} - \frac{\partial T^{**}}{\partial q_1'} \frac{\partial N}{\partial q} \right] + \frac{\partial N}{\partial q_1} (T^{**} - U - \mu) = 0 \end{aligned} \tag{4.7}$$

In view of the condition (4.3), and N = const, we obtain the equation

$$\mu = T^{**} - U$$

Let us consider the solution of the equations of motion (4.4) which passes through the arbitrarily given points $M_0(q^{\circ}, q_1^{\circ})$ and $M_1(q', q_1')$.

For this real motion of systems investigated by Chaplygin, the integral of the kinetic energy $T^{**} - U = h$ is valid. Let us set the multiplier $\mu = h$. According to (4.7), the solution of the equations of motion (4.4) serves also as a solution of Euler's equation (4.5). This solution is an extremal, and depends on the multiplier $\mu = h$. Substituting the extremal into the condition (4.2), we obtain, finally, the sought constant τ_1.

Since the extremal, and the real motion that pass through the points M_0 and M_1 are unique, the real motion must coincide with the extremal, and since the points $M_0(q^{\circ}, q_1^{\circ})$, $M_1(q', q_1')$ are arbitrary, the set of real motions of the nonholonomic system coincides with the set of extremals of the isoperimetric problem (4.1).

This isoperimetric problem can be replaced by the problem on the unconditional extremum of the integral

$$\int_{t_0}^{t_1} F\,dt, \qquad F = (T^{**} + U + h)\,N \tag{4.8}$$

Here, H is the total energy of the real motion. It follows from what has been said that the conditions for the equivalence of Chaplygin's equations (4.4) and Euler's equations of the problem on the extremum of the integral (4.8), can be reduced to Chaplygin's condition (4.3) for the reducing multiplier N.

The results of Chaplygin can thus be obtained by the method of the equations of equivalence of Section 3; they are valid for conservative systems, and for those cases when the equations of equivalence admit a function F (4.8) which does not depend on the velocities q_β' ($\beta = 1, \ldots, m$), and has the special structure $F = (T^{**} + U + h)N$.

Remark. The final equations for equivalence differ from Chaplygin's equations for the function N in so far as that the former ones follow from the condition (4.3) when it is satisfied identically by the velocities q', q_1', but not by impulses, as is the case for Chaplygin's equations. This difference removes the restrictions on the kinetic energy and on the constraints, which have to be imposed in Chaplygin's method for more than two freeecoordinates. This fact was pointed out by

M.I. Efimov.*

5. Let us now consider the second class of nonholonomic systems. Suppose that the Lagrange coordinates of the system are subject to only one equation of constraint

$$\omega = q_1' + \sum_{r=1}^{n-1} a_{1+r} q_{1+r}' + a = 0, \qquad a \neq 0 \tag{5.1}$$

while Lagrange's function of the system, and the equation of constraint are independent of time. Then, as was shown in Section 3, one has to supplement the equations of equivalence (3.2) by the equations of equivalence (3.3)

$$R_{k,\,1+r,\,1+\rho,\,1+\tau} = 0 \qquad (k = 1, \ldots, n;\ r,\ \rho,\ \tau = 1, \ldots, n-1) \tag{5.2}$$

We will prove that for the identical satisfaction of Equations (5.2), it is necessary and sufficient that the equation

$$\delta q_1 + \sum_{r=1}^{n-1} a_{1+r} \delta q_{1+r} = 0 \tag{5.3}$$

which determines the "possible displacements" of the system, be holonomic.

Let us now rewrite the function R_k in terms of the determinants Δ_k and Δ (Section 2). Using the previous notation, we obtain

$$\Delta_{k,\,1+r,\,1+\rho,\,1+\tau} = 0 \tag{5.4}$$

Here $\Delta_{k,1+r,1+\rho,1+\tau}$ is obtained from the determinant Δ by replacing the elements of the kth column by the coefficients of the third order terms in $q_{1+r}' q_{1+\rho}' q_{1+\tau}'$. It is easy to show [2] that under the conditions $\Delta \neq 0$, and (5.4), this column is proportional to the last column of the determinant Δ, which consists of the coefficients a_s of the equation of constraint. We note that the terms of the third order enter only into the products

$$\lambda \left(a_s' - \frac{\partial \omega}{\partial q_s} \right) \qquad (s = 1, \ldots, n)$$

The conditions for the proportionality of the columns is, therefore

$$-\frac{\partial a_s}{\partial q_1} a_{1+r} + \frac{\partial a_s}{\partial q_{1+r}} - \frac{\partial a_{1+r}}{\partial q_s} = k_{1+r} a_s \tag{5.5}$$

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* Efimov, M.I., On the equations of Chaplygin for nonholonomic systems and the method of a reducing multiplier. Dissertation. Institute of Mechanics, Akad. Nauk SSSR, 1953.

The coefficients of proportionality k_{1+r} are included in the b_{sk} because of λ_k (2.14). From the first ($s = 1$) Equation (5.5) we obtain the coefficient $k_{1+r} = -\partial a_{1+r}/\partial q_1$, and substituting it in the remaining ($s = 1 + \rho$, $\rho = 1, \ldots, n - 1$) Equations (5.5), we obtain the equations

$$\frac{\partial a_{1+r}}{\partial q_1} a_{1+\rho} - \frac{\partial a_{1+\rho}}{\partial q_1} a_{1+r} + \frac{\partial a_{1+\rho}}{\partial q_{1+r}} - \frac{\partial a_{1+r}}{\partial q_{1+\rho}} = 0 \qquad (r, \rho = 1, \ldots, n-1)$$

which imply that Equation (5.3) is holonomic.

The necessity is thus established. By carrying out the arguments in the reverse order, one can show that the condition is sufficient. Thus we obtain a theorem analogous to Theorem 3.1.

Theorem 5.1. Let a conservative system with a nonholonomic constraint (5.1) be given. Suppose that there exists for it a solution of the equation of equivalence (3.2), and suppose that Equation (5.2), which determines the "possible displacements" of the system, is holonomic. Then every real motion, in the class of all kinematically admissible motions of the system which satisfy the boundary conditions

$$q_s(t_0) = q_{s0}, \qquad q_s(t) = q_{s1} \qquad (s = 1, \ldots, n)$$

will satisfy the equation

$$\delta \int_{t_0}^{t} F\, dt = 0$$

where the upper limit of the integral is a free variable.

6. Suppose that the state of holonomic systems is determined by the coordinates $q_1, \ldots, q_n$ subject to the auxiliary integrable constraints (1.1). Suppose that these holonomic systems have a unique Lagrange function (constructed without taking into account the auxiliary constraints $\omega_\beta = 0$), and different equations $\omega_\beta = 0$. Then the motions of these systems are described by one and the same variational problem, for example, by Hamilton's principle

$$\delta \int_{t_0}^{t_1} L\, dt = 0 \qquad \text{when } \omega_\beta = 0$$

On the other hand, for different arbitrary differential equations of constraint (1.1), the integral functions F of the problem (1.3) which are formed by the solutions of the equations of equivalence, are generally distinct.

We shall say that the integrand F of the conditional variational problem does not depend on the equations of constraint if the corresponding equations of equivalence do not depend on the coefficients $a_{\beta, m+r}$, a_β

of the equation of constraint, and on their derivatives

$$\frac{\partial a_{\beta,\, m+r}}{\partial q_s}, \qquad \frac{\partial a_\beta}{\partial q_s}$$

In other words, the equations of constraint imply that the following equations are satisfied

$$\frac{\partial}{\partial a_{\beta,\, m+r}}(R_k - \Phi_k) = 0, \qquad \frac{\partial}{\partial a_\beta}(R_k - \Phi_k) = 0 \tag{6.1}$$

$$\frac{\partial}{\partial \xi}(R_k - \Phi_k) = 0, \qquad \frac{\partial}{\partial \eta}(R_k - \Phi_k) = 0, \qquad \xi = \frac{\partial a_{\beta,\, m+r}}{\partial q_s}, \qquad \eta = \frac{\partial a_\beta}{\partial q_s}$$

$$(k,\ s = 1, \ldots, n;\ \beta = 1, \ldots, m;\ r = 1, \ldots, n-m) \tag{6.2}$$

Theorem 6.1. In order that the integrand of the variational problem for a material system may be independent of the equations of constraint, it is necessary and sufficient that these equations be holonomic.

Proof. The sufficiency is established by the fact that Hamilton's principle is valid for holonomic systems.

Let us prove the necessity. Suppose that the function F does not depend on the equations of constraint, and, hence, that Equations (6.1) and (6.2) are satisfied. Let us turn our attention to Equations (6.2). The functions Φ_k do not depend on $\xi = \partial a_{\beta,\, m+r}/\partial q_s$, $\eta = \partial a_\beta/\partial q_s$. Hence, the equations of constraint imply that

$$\frac{\partial R_k}{\partial \xi} = 0, \qquad \frac{\partial R_k}{\partial \eta} = 0 \tag{6.3}$$

In place of the functions R_k in Equations (6.3) one can always first take the functions Q_k of Section 2 (the functions R_k are obtained from the functions Q_k through the exclusion of the multipliers λ_β). We thus have in place of (6.3) the equations

$$\frac{\partial Q_k}{\partial \xi} = 0, \qquad \frac{\partial Q_k}{\partial \eta} = 0 \tag{6.4}$$

which are satisfied because of the equations of constraint. The derivatives $\xi = \partial a_{\beta,\, m+r}/\partial q_s$, $\eta = \partial a_\beta/\partial q_s$ enter into the functions Q_k only through the products

$$\sum_{\beta=1}^{m} \lambda_\beta \left(a'_{\beta s} - \frac{\partial \omega_\beta}{\partial q_s} \right)$$

Therefore, Equations (6.4) and the equations

$$\frac{\partial Q_k}{\partial \lambda_\beta} = 0 \qquad (k = 1, \ldots, n;\ \beta = 1, \ldots, m) \tag{6.5}$$

will arise and vanish simultaneously. In accordance with [2], it follows from this that the equations of constraint are holonomic. Hereby, Equations (6.1) will be satisfied identically.

Remark. The theorem shows that the variational principles, which are known for holonomic systems, are not valid for nonholonomic systems.

7. *Examples.* a) The motion of an automobile under inertia [3]. The position of the automobile can be determined by four coordinates

$$q_1 = x, \quad q_2 = y, \quad q_3 = \alpha, \quad q_4 = \theta$$

where x and y are the coordinates of the center of inertia of the automobile on a horizontal plane of the motion; α is the angle formed by its longitudinal axis with the x-axis; θ is the angle between the longitudinal axis of the automobile and the straight line which connects the middle of the front axis of the automobile with its center of rotation. The angle θ characterizes the turn of the steering wheel.

Let the angle of turning of the steering wheel be given: $\theta = \theta(t)$. Furthermore, we have two nonholonomic equations of constraints (when $\theta \neq 0$)

$$\begin{aligned} x' &= (l \cot\theta \cos\alpha - a\sin\alpha)\alpha' = -a_{13}\,\alpha' \\ y' &= (l\cot\theta \sin\alpha + a\cos\alpha)\,\alpha' = -a_{23}\,\alpha' \end{aligned} \tag{7.1}$$

Here l is the length of the automobile, a is the distance from the center of inertia to the rear axis of the automobile. Neglecting the mass of the wheels, we have the following expression for the kinetic energy of the automobile

$$T = \frac{m}{2}(x'^2 + y'^2) + \frac{1}{2} I\alpha'^2 \qquad (I \text{ is the central moment of inertia})$$

The equations of motion of the automobile that moves under inertia have the form*

$$\begin{aligned} mx'' &= -\mu_1 \sin\alpha - \mu_2 \sin(\alpha + \theta) \\ my'' &= \mu_1 \cos\alpha + \mu_2 \cos(\alpha + \theta) \\ I\alpha'' &= -\mu_1 a + \mu_2 (l - a)\cos\theta \end{aligned} \tag{7.2}$$

Here,

$$\mu_1 = m \cot\theta \left[(l - a)\,\alpha'^2 + \frac{l\,(al - \rho^2)\,\alpha'\theta'}{\rho^2 \sin^2\theta + l^2\cos^2\theta}\right], \qquad \rho^2 = \frac{I + ma^2}{m}$$

* Novoselov, V.S., Some questions of nonholonomic mechanics. Dissertation. MGU (Moscow State University), 1958.

$$\mu_2 = \frac{m}{\sin\theta}\left[a\alpha'^2 + \frac{\rho^2 l\alpha'\theta'}{\rho^2\sin^2\theta + l^2\cos^2\theta}\right]$$

Chaplygin's theorem is applicable to the problem under consideration because the system is not conservative. For the construction of the equations of equivalence one uses only the last equation. Here,

$$\Phi_{30} = 0, \qquad \Phi_{33} = \frac{l^2\cot\theta}{\rho^2\sin^2\theta + l^2\cos^2\theta}, \qquad \Phi_{333} = 0$$

We will look for the function F in the form

$$F = \frac{1}{2}(b_{11}x'^2 + b_{22}y'^2 + b_{33}\alpha'^2 + 2b_{12}x'y' + 2b_{13}x'\alpha' + 2b_{23}y'\alpha')$$

with coefficients that depend only on θ. Because of (2.9),

$$\lambda_1 = -b_{11}x' - b_{12}y' - b_{13}\alpha', \qquad \lambda_2 = -b_{12}x' - b_{22}y' - b_{23}\alpha'$$

Solving the equations of the extremals for α'', we obtain

$$R_{30} = 0, \qquad R_{33} = -\frac{1}{\Delta}\frac{d\Delta}{d\theta}$$

$$R_{333} = \frac{1}{\Delta}\left[\frac{\partial a_{13}}{\partial\alpha}(b_{13} - a_{13}b_{11} - a_{23}b_{12}) + \frac{\partial a_{23}}{\partial\alpha}(b_{23} - a_{23}b_{22} - a_{13}b_{12})\right]$$

where

$$\Delta = a_{13}^2 b_{11} + a_{23}^2 b_{22} + b_{33} + 2a_{13}a_{23}b_{12} - 2a_{13}b_{13} - 2a_{23}b_{23}$$

We have two equations of equivalence

$$R_{333} = 0 \quad \frac{1}{\Delta}\frac{d\Delta}{d\theta} = -\frac{l^2\cot\theta}{\rho^2\sin^2\theta + l^2\cos^2\theta} \tag{7.3}$$

The first equation of (7.3) will be satisfied identically when $b_{11} = b_{22}$, $b_{12} = b_{13} = b_{23} = 0$. Solving, next, the second equation of (7.3), we find

$$\Delta = m\sqrt{\rho^2 + l^2\cot^2\theta} = b_{11}(l^2\cot^2\theta + a^2) + b_{33} \tag{7.4}$$

Selecting $b_{33} = b_{11}(\rho^2 - \alpha^2)$, we obtain $b_{11} = m/\sqrt{(\rho^2 + l^2\cot^2\theta)}$ and

$$F = \frac{1}{2}\frac{m}{\sqrt{\rho^2 + l^2\cot^2\theta}}\left(x'^2 + y'^2 + \frac{I}{m}\alpha'^2\right) \tag{7.5}$$

Furthermore, by solving Euler's equation for the variational problem with the function F (7.5), and with the equation (7.1), for x'', y'', α'', and substituting

$$\lambda_1 = -\frac{mx'}{\sqrt{\rho^2 + l^2\cot^2\theta}}, \qquad \lambda_2 = -\frac{my'}{\sqrt{\rho^2 + l^2\cot^2\theta}}$$

we obtain the accelerations of the real motion.

If one considers the problem in the space α, then

$$b_{11} = b_{22} = 0, \qquad \Delta = b_{33} = m\sqrt{\rho^2 + l^2 \cot^2\theta}, \qquad F = \frac{m}{2}\sqrt{\rho^2 + l^2\cot^2\theta}\,\alpha'^2$$

b) Suppose that the position of the system is determined by two Lagrange coordinates restricted by nonholonomic equations of constraints $\omega = q_1' - q_2' - (q_1 + q_2) = 0$, and suppose that the kinetic energy of the system is $T = (q_1'^2 + q_2'^2)$, while the force function $U = 0$. Then the equation of motion of Routh will yield

$$q_1'' = \tfrac{1}{2}(q_1' + q_2'), \qquad q_2'' = -\tfrac{1}{2}(q_1' + q_2')$$

We are looking for a function F of the form

$$F = \tfrac{1}{2}(b_{11}q_1'^2 + 2b_{12}q_1'q_2' + b_{22}q_2'^2)$$

Then the multiplier $\lambda = -F/(q_1 + q_2)$, in view of Equation (2.14). It is easy to give a particular solution of the corresponding equations of equivalence: $b_{sk} = (q_1 + q_2)c_{sk}$, where c_{sk} are constants connected by the relation $c_{11} - 2c_{12} + c_{22} = 0$ (which implies that $c_{12} \neq 0$, $c_{11} \neq c_{22}$).

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